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Endogenous Growth in an OLG Model with a Fixed Factor

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Abstract

This paper examines the conditions for endogenous growth in an overlapping generations (OLG) model with two sectors of production when the returns to scale may be non constant and provides the technological conditions for the highest income growth rate.

Keywords: endogenous growth, increasing returns, human capital, land, overlapping generations.

JEL Classification numbers: E13, O11, O41.

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1 Introduction

In two-sector growth models, constant returns to scale technology applying to both sectors implies long-term growth when the reproducible factors grow at the same rate.[1] The structure of these models is generally the same. One of the two sectors produces the physical capital and the other sector produces an investment good that varies across models. For example, this good may be human capital as in Lucas (1988), the number of varieties of products as in Romer (1990) or the quality of products as in Aghion and Howitt (1992). Both sectors may use the same inputs according to technologies that may be different.

This paper considers a two-sector model with overlapping generations, in which the investment goods are physical and human capital. The production of the final good and the accumulation of human capital both uses quantities of physical and human capital as inputs in proportions that are assumed to be exogenous as in Rebelo (1991). In standard models, the returns to scale are constant in the output and the education sectors and the growth rates of both sectors are equal. The level of the steady state income growth rate depends on the fractions of physical and human capital allocated to both sectors and the income shares of both investment goods.

More general specifications of the production functions for the final good and human capital have been considered in the literature in order to study situations with increasing returns. Mulligan and Sala-I-Martin (1993) study combinations of non-constant returns to scale technologies in a neoclassical growth model that are consistent with balanced growth in the long run. It is even possible to obtain increasing returns to scale in both production functions if a non-reproducible factor such as land or raw labor (or a combination of the two) is added to the output technology for example, and remain in a competitive framework. This is the modeling approach we adopt in this paper and apply it to a growth model with overlapping generations living for two periods. we consider a general specification for production functions in the sense that we use Cobb-Douglas functional forms in which the income shares of inputs may not sum up to one. Our objective is to identify the technological conditions yielding the highest income growth rate. After determining the balanced growth condition along the stationary path, we compare the growth rates yielded by the possible combinations of non-constant returns to scale technologies that verify it, and give the conditions leading to the fastest steady state growth rate.

The paper is organized as follows. Section 2 defines the model. Section 3 determines the balanced growth condition. Section 4 studies the growth rates along the balanced growth path. Section 5 presents an example derived from the Uzawa-Lucas model. Finally, section 6 concludes.

[1]See Barro and Sala-I-Martin (1995) for a general treatment of two-sector growth models

2 The model

The model is an extension of the overlapping generations model of Allais (1947) and Diamond (1965). The economy is closed and populated by overlapping generations, each living for three periods. The generation in period t is populated by N_t households and the total population grows at a constant rate n. When young, the households work, consume and invest a part of their income in both physical human capital which are rented and used by the firms in the next period. When old, they consume the return of their savings and die. In addition, each household owns a piece of land and a share of the firms. They thus receive land rents and profits. As there is no land market, they transmit their property rights over land to their children when they are old. As a result, only the working generation owns land. Each household is owner of the firms and receives interest on the capital rental. The firms buy inputs and produce the same single good in perfectly competitive markets. The single final good produced in this economy can either be consumed by the adult and the old generations or accumulated by the young households as capital for future production. The only consumption good is measured in units of final output.

2.1 Production technology

At each period the representative firm at the aggregate level produces a single good under a technology with constant or non-constant (social) returns to scale. There are three factors of production: physical capital, human capital and land. We assume that the production function of the representative firm is given by

$$Y_t = A(vK_t)^{\alpha}(uH_t)^{\gamma}L^{\mu}, \quad 0 \leqslant \alpha, \gamma, \mu, u, v \leqslant 1 \tag{1}$$

where Y_t is the output, K_t is physical capital, H_t the stock of human capital used by the representative firm at time t, and $A > 0$ is a technological parameter. Physical capital is assumed to be fully depreciated after one period. L is the land endowment of this economy, which is assumed to be fixed over time and to enter the aggregate production function.[2] Production uses a fraction v of physical capital and a fraction u of human capital. The parameters α, γ and μ are the income shares or productivity elasticities of physical capital, human capital and land respectively. Each of these parameters are assumed to be strictly positive and strictly smaller than one. The problem of the firm is to maximize profits. At time t, the firm inherits the stock of physical capital from the previous period and has to decide upon the quantity of effective labor to hire. Therefore, an interior solution corresponding to a maximum of profits exists if the production function is concave, i.e., if the returns to scale with respect to the reproducible factors are non increasing:

$$\alpha + \gamma \leqslant 1 \tag{2}$$

[2]The fixed factor could also be, for instance, raw labor or a combination of land and raw labor.

2.2 Human capital

Human capital is assumed to be a productive and a reproducible asset, whose accumulation results from a technology that may be different from the one applying to physical capital. Thus human capital may be an imperfect substitute for consumption. Like physical capital, it is assumed that human capital is homogenous. The production function for the human capital accumulation is defined by

$$H_{t+1} = B[(1-v)K_t]^{\theta}[(1-u)H_t]^{\eta}, \quad 0 \leqslant \theta, \eta \leqslant 1, \tag{3}$$

where $B > 0$ is a technological parameter, θ and η are the elasticities of human capital accumulation with respect to the past stocks of physical and human capital respectively. The returns to scale of human capital accumulation are decreasing if $\theta + \eta < 1$, constant if $\theta + \eta = 1$, and increasing if $\theta + \eta > 1$. The stock of human capital at time $t+1$ is assumed to depend on the inherited stocks of physical and human capital, which are imperfect substitutes. If the fraction of the human capital input were higher than $1-u$, then there would be a positive externality in the sector of the human capital production. In other words, social returns in the sector of human capital accumulation would be higher than private returns. We assume that human capital fully depreciates after one period. If $\theta < \alpha$, the sector of human capital accumulation is relatively intensive in human capital while the output sector is relatively intensive in physical capital. This is the case in Uzawa (1965) and Lucas (1988) where $\theta = 0$ and $\alpha > 0$. An interior solution for an optimal choice of saving is obtained if the private marginal return to investment in physical capital is decreasing, i.e.:

$$\theta \leqslant 1 \tag{4}$$

2.3 Preferences

The representative consumer maximizes a logarithmic utility function of the type

$$U(c_t, d_{t+1}) = \ln c_t + \beta \ln d_{t+1} \tag{5}$$

subject to the following budget constraint,

$$\begin{aligned} c_t + s_t &= w_t \frac{uH_t}{N_t} + \frac{\pi_t L}{N_t} \\ d_{t+1} &= R_{t+1} s_t \end{aligned}$$

Utility depends on consumption when young c_t and on consumption when old d_{t+1}. The parameter $\beta > 0$ is the psychological discount factor. The adults supply inelastically one unit of labor and earn $w_t u H_t$, where w_t is the wage per unit of human capital and H_t is the aggregate level of human capital. They also receive $\pi_t L/N_t$ as land rent. Their income is allocated to consumption and saving, s_t, for future consumption. When old agents spend all their saving and accrued interest on consumption. R_{t+1} is the interest factor.

2.4 Profits

The maximization problem of the representative firm is defined by

$$\{K_t, H_t\} = \arg\max \{A(vK_t)^{\alpha}(uH_t)^{\gamma}L^{\mu} - w_t uH_t - R_t vK_t - \pi_t L\}.$$

The representative firm maximizes its profits subject to the constraint of technology. Therefore, these profits depend on the technology. When returns to scale (social returns) are non-constant, profits are different from zero. In this case, we assume that (positive or negative) profits are redistributed to land owners. Therefore, π_t will represent the remuneration of the land factor and also the residual share in output (externality). Depending on the technology, the externality can be positive, negative or null.

2.5 Optimal behaviors

The representative consumer-producer chooses optimally c_t, d_{t+1} and H_t. As a representative firm, he hires the human capital input, uH_t, according to (1). The human capital accumulates according to (2). As a representative consumer, he chooses c_t, d_{t+1} and therefore, s_t, according to (4). Since profits reach a maximum by the concavity of the production function, the production factors are paid at their marginal productivities. Hence, the first order necessary conditions of the firm's program (1) are:

$$R_t = \alpha A(vK_t)^{\alpha-1}(uH_t)^{\gamma}L^{\mu}, \tag{6}$$

$$w_t = \gamma A(vK_t)^{\alpha}(uH_t)^{\gamma-1}L^{\mu}, \tag{7}$$

The young adult land owners receive the land rent equal to the marginal productivity of this factor and the residual income share:

$$\pi_t = (1-\alpha-\gamma)A(vK_t)^{\alpha}(uH_t)^{\gamma}L^{\mu-1}. \tag{8}$$

The marginal productivity of land, $\frac{\partial Y}{\partial L}$, is positive. As for the residual income share, $\pi_t - \frac{\partial Y}{\partial L}$, it is negative when returns to scale are increasing ($\alpha+\gamma+\mu > 1$), it is positive when they are decreasing ($\alpha+\gamma+\mu < 1$) and it is null when they are constant ($\alpha+\gamma+\mu = 1$). The first order necessary conditions of the consumer's program (4) are:

$$s_t = \frac{\beta}{1+\beta}\left(w_t \frac{uH_t}{N_t} + \frac{\pi_t L}{N_t}\right). \tag{9}$$

Saving is thus a function of the labor income and land rent.

2.6 Equilibrium

The equilibrium on the good market at period t is given by the national accounting identity:

$$Y_t = N_t c_t + I_t, \tag{10}$$

where $N_t c_t$ is the aggregate consumption at period t. The aggregate investment I_t is equal to the future physical capital stock K_{t+1} since the current capital stock K_t fully depreciates at the end of the current period. The equilibrium on the capital market derives from (10) and yields:

$$K_{t+1} = N_t s_t, \tag{11}$$

where $N_t s_t$ is the aggregate saving at period t.

Equilibrium requires that physical capital receive the same return in both sectors. The same condition must apply for human capital. This leads to the following relation between u and v:

$$\frac{\alpha}{\gamma}\left(\frac{u}{1-u}\right) = \frac{\theta}{\eta}\left(\frac{v}{1-v}\right). \tag{12}$$

It can be shown that $\frac{dv}{du} > 0$.

The dynamics will be analyzed in terms of three stationary variables: the physical-human capital ratio k_{t+1}, the growth factor of human capital $x_{t+1} = H_{t+1}/H_t$, and the growth factor of the economy $g_{t+1} = Y_{t+1}/Y_t$. Equilibrium requires a stationary physical-human capital ratio that can be derived from the equilibrium interest factor (equation 6):

$$k_t \equiv \frac{K_t}{H_t^{\frac{\gamma}{1-\alpha}}}. \tag{13}$$

The marginal productivity of the production factors can be rewritten as:

$$\begin{aligned}
R_t &= \alpha A v^{\alpha-1} u^{\gamma} k_t^{\alpha-1} L^{\mu} \\
w_t &= \gamma A v^{\alpha} u^{\gamma-1} k_t^{\alpha} H_t^{\frac{\gamma}{1-\alpha}-1} L^{\mu} \\
\pi_t &= (1-\alpha-\gamma) A v^{\alpha} u^{\gamma} k_t^{\alpha} H_t^{\frac{\gamma}{1-\alpha}} L^{\mu-1}
\end{aligned}$$

An equilibrium can now be characterized as follows: given initial conditions $\{K_0, H_0\}$ satisfying (11), an equilibrium is a vector of positive quantities $(K_t, H_t, c_t, d_t, s_t, \pi_t)_{t\geqslant 0}$ and prices $(R_t, w_t)_{t\geqslant 0}$ such that equations (1) to (13) hold. Equations (1) to (13) can be reduced to a system of three non-linear difference equations of the first order, describing the dynamics of the physical-human capital ratio k_t, the growth factor of human capital accumulation x_t and the growth factor of the economy g_t:

$$k_{t+1} = \frac{\beta(1-\alpha)Av^{\alpha}u^{\gamma}L^{\mu}}{(1+\beta)B^{\frac{\gamma}{1-\alpha}}(1-v)^{\frac{\theta\gamma}{1-\alpha}}(1-u)^{\frac{\eta\gamma}{1-\alpha}}} k_t^{\alpha-\frac{\theta\gamma}{1-\alpha}} H_t^{\frac{\gamma}{1-\alpha}\left(1-\eta-\frac{\theta\gamma}{1-\alpha}\right)} \tag{14}$$

$$x_{t+1} = B(1-v)^{\theta}k_t^{\theta}(1-u)^{\eta}H_t^{-1+\eta+\frac{\theta\gamma}{1-\alpha}} \tag{15}$$

$$g_{t+1} = \frac{k_{t+1}^{\alpha}}{k_t^{\alpha}}(x_{t+1})^{\frac{\gamma}{1-\alpha}} \tag{16}$$

Equation (14) gives the dynamics of the physical-human capital ratio, equation (15) the growth factor of the human capital stock and equation (16) the growth factor of the economy. The growth rate of the income per capita is

$$\rho_{t+1} = \frac{y_{t+1}}{y_t} = \frac{1}{1+n}\frac{Y_{t+1}}{Y_t} = \frac{g_{t+1}}{1+n}, \tag{17}$$

where y_t is the income per capita at period t.

In the rest of the paper, we want to analyze growth paths of this economy using different combinations of technology applied to both sectors of production.

3 Balanced growth condition

Equation (16) gives the growth factor of the economy and allows us to derive the balanced growth condition.

Proposition 1 [3]

A two-sector OLG model with a fixed factor, exhibiting non-constant returns to scale technologies, admits a balanced growth path for conditional values.

Proof:

A balanced growth path exists if and only if the growth factor of the economy g_{t+1} of equation (16) is equal to a constant, which is the case if the growth factor of human capital x_{t+1} of equation (15) is also equal to a constant. This requires that

[3]This proposition is owed to Mulligan and Sala-I-Martin (1993).

$$\frac{\theta\gamma + \eta(1-\alpha) - 1 + \alpha}{1-\alpha} = 0$$

i.e.,

$$\frac{\gamma}{1-\alpha} = \frac{1-\eta}{\theta}. \tag{18}$$

Equation (18) allows for three cases:

i) $\alpha + \gamma < 1$ and $\theta + \eta > 1$;

ii) $\alpha + \gamma = 1$ and $\theta + \eta = 1$;

iii) $\alpha + \gamma > 1$ and $\theta + \eta < 1$.

Case iii) is ruled out by assumption (2) since an interior solution for the optimizing firm requires that private returns to reproducible factors (K and H) must be lower or equal to one. Therefore, the balanced growth condition (18) allows for two possible growth regimes: case 1), in which there are increasing returns to scale in production and human capital accumulation; and case 2), in which production exhibit increasing returns to scale and human capital accumulation exhibits constant returns to scale. Interestingly, condition (18) can be interpreted in terms of growth rates of the reproducible factors. In fact, along the balanced growth path,

$$\frac{Y_{t+1}}{Y_t} = \left(\frac{K_{t+1}}{K_t}\right)^{\alpha} \left(\frac{H_{t+1}}{H_t}\right)^{\gamma}.$$

Since the capital stock K is equal to aggregate saving, i.e. a fraction of the output Y, then along the balanced growth path, the income growth factor is equal to the growth factor of the physical capital. This implies,

$$g_{t+1} = (x_{t+1})^{\frac{\gamma}{1-\alpha}}. \tag{19}$$

If $\gamma < 1-\alpha$, i.e., if returns to scale to reproducible factors in the production function are decreasing, the growth rate of human capital accumulation is higher than the growth rate of the economy and lower otherwise.[4] In the sector of the production of human capital, the growth rate of the human capital stock must be constant along the balanced growth path and yields

[4] It could be possible to have increasing returns to scale (social returns) to reproducible factors in the production function ($\alpha + \gamma > 1$) in a setting with knowledge spillovers that firms could not internalize. Therefore, private returns to reproducible factors would be constant while social returns would be increasing. In our setting, firms internalize all returns.

$$\left(\frac{H_{t+1}}{H_t}\right)^{1-\eta} = \left(\frac{K_{t+1}}{K_t}\right)^{\theta}.$$

Since the capital stock K is equal to aggregate saving, i.e., a fraction of the output Y, then, along the balanced growth path,

$$g_{t+1} = (x_{t+1})^{\frac{1-\eta}{\theta}}. \tag{20}$$

Equality (20) is not necessarily the same as equality (19). The condition for a balanced growth path (18) states that the growth rates of physical capital and human capital must be identical in the output and human capital production sectors.

3.1 Two balanced growth regimes

The balanced growth condition (18) allows to consider two growth regimes for an economy with land and human capital. The first corresponds to increasing returns to scale in both production and human capital accumulation. In the second regime, returns to scale are increasing in production and constant in human capital accumulation.

3.1.1 Regime 1: increasing returns to scale in production and human capital accumulation

In this growth regime, we consider an OLG model with land exhibiting increasing returns to scale in output and human capital technologies. The production technology of the firm is defined by:

$$Y_t = A(vK_t)^{\alpha}(uH_t)^{\gamma}L^{\mu},$$

where the elasticities of the reproducible factors $\alpha + \gamma < 1$ and $\mu > 0$. However, the sum of the factor elasticities $\alpha + \gamma + \mu$ can be higher than 1, which yields increasing returns to scale in production. Human capital accumulates according to

$$H_{t+1} = B[(1-v)K_t]^{\theta}[(1-u)H_t]^{\eta}, \quad 0 < \theta < 1.$$

where it is assumed that $\theta + \eta > 1$. The returns to scale of human capital accumulation are thus increasing. Equilibrium requires a stationary physical-human capital ratio:

$$k_t \equiv \frac{K_t}{H_t^{\frac{\gamma}{1-\alpha}}}.$$

Marginal productivities of production factors are as follows:

$$
\begin{aligned}
R_t &= \alpha A v^{\alpha} u^{\gamma} k_t^{\alpha-1} L^{\mu} \\
w_t &= \gamma A v^{\alpha} u^{\gamma} k_t^{\alpha} H_t^{\frac{\gamma}{1-\alpha}-1} L^{\mu} \\
\pi_t &= (1-\alpha-\gamma) A v^{\alpha} u^{\gamma} k_t^{\alpha} H_t^{\frac{\gamma}{1-\alpha}} L^{\mu-1}
\end{aligned}
$$

The variables R_t, w_t, π_t are the equilibrium factor prices per unit of inputs K_t, H_t and L. The system of three non-linear difference equations (14)-(16) admits a balanced growth path if condition (18) is satisfied. Then, the system can be rewritten as

$$
\begin{aligned}
k_{t+1} &= \frac{\beta(1-\alpha) A v^{\alpha} u^{\gamma} L^{\mu}}{(1+\beta) B^{\frac{\gamma}{1-\alpha}} (1-v)^{\frac{\theta\gamma}{1-\alpha}} (1-u)^{\frac{\eta\gamma}{1-\alpha}}} k_t^{\alpha-1+\eta} && (21) \\
x_{t+1} &= B(1-v)^{\theta}(1-u)^{\eta} k_t^{\theta} && (22) \\
g_{t+1} &= \frac{k_{t+1}^{\alpha}}{k_t^{\alpha}} (x_{t+1})^{\frac{\gamma}{1-\alpha}} && (23)
\end{aligned}
$$

Along the balanced growth path, the stock of physical capital per effective unit of labor and the growth factor of the economy are constant:

$$
\begin{aligned}
\bar{k}_1 &= \left(\frac{\beta(1-\alpha) A v^{\alpha} u^{\gamma} L^{\mu}}{(1+\beta) B^{\frac{\gamma}{1-\alpha}} (1-v)^{\frac{\theta\gamma}{1-\alpha}} (1-u)^{\frac{\eta\gamma}{1-\alpha}}} \right)^{\frac{1}{2-\alpha-\eta}} && (24) \\
\bar{x}_1 &= [B(1-v)^{\theta}(1-u)^{\eta}]^{1-\frac{\theta\gamma}{(1-\alpha)(2-\alpha-\eta)}} \left(\frac{\beta(1-\alpha) A v^{\alpha} u^{\gamma} L^{\mu}}{1+\beta} \right)^{\frac{\theta}{2-\alpha-\eta}} && (25) \\
\bar{g}_1 &= (x_1)^{\frac{\gamma}{1-\alpha}} && (26)
\end{aligned}
$$

3.1.2 Regime 2: Increasing returns to scale in production and constant returns to scale in human capital accumulation

In regime 2, we consider an OLG model with land, in which the sum of the elasticities of the reproducible factors in the production technology and in human capital accumulation sum up to one. The production technology of the firm is defined by:

$$
Y_t = A(vK_t)^{\alpha}(uH_t)^{1-\alpha} L^{\mu},
$$

where the elasticities of the reproducible factors $\alpha + \gamma = 1$ and $\mu > 0$. However, the sum of the factor elasticities, $\alpha + \gamma + \mu > 1$, yields increasing returns to scale in production. The accumulation technology for human capital is

$$H_{t+1} = B[(1-v)K_t]^{\theta}[(1-u)H_t]^{1-\theta}, \quad 0 < \theta < 1.$$

where $\eta = 1 - \theta$. The returns to scale of human capital accumulation are thus constant. Equilibrium requires a stationary physical-human capital ratio:

$$k_t \equiv \frac{K_t}{H_t^{\frac{\gamma}{1-\alpha}}} = \frac{K_t}{H_t}.$$

Marginal productivities of production factors are as follows:

$$\begin{aligned} R_t &= \alpha A v^{\alpha} u^{1-\alpha} k_t^{\alpha-1} L^{\mu} \\ w_t &= (1-\alpha) A v^{\alpha} u^{1-\alpha} k_t^{\alpha} L^{\mu} \\ \pi_t &= 0 \end{aligned}$$

The marginal productivity of land is positive but exactly offset by the negative residual income share. Therefore, the return to land is null. The system of two non-linear difference equations (14)-(16) becomes:

$$k_{t+1} = \frac{\beta(1-\alpha) A v^{\alpha} u^{1-\alpha} L^{\mu}}{(1+\beta) B (1-v)^{\theta} (1-u)^{1-\theta}} k_t^{\alpha-\theta} \tag{27}$$

$$x_{t+1} = B(1-v)^{\theta}(1-u)^{1-\theta} k_t^{\theta} \tag{28}$$

$$g_{t+1} = \frac{k_{t+1}^{\alpha}}{k_t^{\alpha}} x_{t+1} \tag{29}$$

Equations (28) and (29) show that the system (27)-(29) admits a balanced growth path when k reaches the stationary state. Since the elasticities of the reproducible factors sum up to one and the returns to human capital accumulation are constant ($\theta = 1 - \eta$), per capita income grows linearly. Along the balanced growth path, the stock of physical capital and the growth rate of the economy are positive constants:

$$\bar{k}_2 = \left(\frac{\beta(1-\alpha) A v^{\alpha} u^{1-\alpha} L^{\mu}}{(1+\beta) B (1-v)^{\theta} (1-u)^{1-\theta}} \right)^{\frac{1}{1-\alpha+\theta}} \tag{30}$$

$$\bar{x}_2 = [B(1-v)^{\theta}(1-u)^{1-\theta}]^{1-\frac{\theta}{1-\alpha+\theta}} \left(\frac{\beta(1-\alpha) A v^{\alpha} u^{1-\alpha} L^{\mu}}{1+\beta} \right)^{\frac{\theta}{1-\alpha+\theta}} \tag{31}$$

$$\bar{g}_2 = \bar{x}_2 \tag{32}$$

4 Analysis of the growth rates along the balanced growth path

The parameters u and v and the presence of a fixed factor L in the production function (1) allow for a rich set of technological combinations in this two-sector OLG model. The parameters u and v characterize exogenously the relative intensity in inputs of the two sectors. When one of the sectors represents the production of human capital it is generally considered that this sector is relatively more intensive in human capital than in physical capital. In other contexts, it is plausible to assume either $u \leqslant v$ or $u > v$. When a fixed factor is included in the model, it is possible to have non-constant returns to scale technologies. The set of technological combinations is defined by equation (12). The objective is now to identify the technological combination generating the highest growth rate along the balanced growth path in both growth regimes and then compare them.

4.1 Condition for the maximum growth rate

The growth rates $\bar{g}_1$ and $\bar{g}_2$ are functions of u, v, α, β, θ and η. The condition for the highest rate is given by the following proposition:

Proposition 2 *In a two-sector OLG model with a fixed factor, the maximum growth rate along the balanced growth path is reached when $u = 1 - \eta$ and $v = \alpha$.*

Proof: see Appendix A.

Since $\theta = 1 - \eta$ in growth regime 2, the condition for the maximum growth rate is $u = \theta$ and $v = \alpha$. All other values for u and$v$ lead to lower growth rates.

4.2 A comparison of growth rates between growth regime 1 and growth regime 2

Once we know the condition for the highest growth rate in regimes 1 and 2, we can compare them.

Proposition 3 *In a two-sector OLG model with a fixed factor, $\bar{g}_1$, the growth rate of regime 1 (increasing returns to scale in production and human capital accumulation), is higher than $\bar{g}_2$, the growth rate of regime 2 (increasing returns to scale in production and constant returns to scale in human capital accumulation), if and only if*

$$B^{1-\alpha} A^{\theta} L^{\theta\mu} < \left(\frac{[(1-\alpha)^{\theta}(1-\phi\theta)]^{1-\alpha}(\beta(1-\alpha)\alpha^{\alpha})^{\theta}}{1+\beta} \right) \left(\frac{\phi\theta}{1-\phi\theta} \right)^{\phi\theta[2(1-\alpha)-\phi\theta]} \tag{33}$$

Proof: see Appendix B.

The difference between the levels of $\bar{g}_1$ and $\bar{g}_2$ ultimately depends on the level of the technological parameters in the production function, A, in the human capital sector, B, and on the level of the fixed factor L. The lower the levels of these parameters, the more likely the growth rate of regime 1 is the highest. If we normalize these parameters to one, then $\bar{g}_1 > \bar{g}_2$.

5 An Example : the Uzawa-Lucas model

Let us now assume that $\theta = 0$, which implies that $v = 1$. This is the framework of Uzawa (1965) and Lucas (1988), in which the accumulation of human capital does not use physical capital as an input. Unlike Lucas (1988), we will further assume that there is no externality in the production of human capital. The production function becomes

$$Y_t = AK_t^{\alpha}(uH_t)^{\gamma}L^{\mu}, \tag{34}$$

and human capital accumulates according to

$$H_{t+1} = B[(1-u)H_t]^{\eta}. \tag{35}$$

The balance growth condition for this model is $\eta = 1$. The system (14)-(16) becomes

$$\begin{aligned} k_{t+1} &= \frac{\beta(1-\alpha)Au^{\gamma}L^{\mu}}{(1+\beta)B^{\frac{\gamma}{1-\alpha}}(1-u)^{\frac{\gamma}{1-\alpha}}}k_t^{\alpha} && (36) \\ x_{t+1} &= B(1-u)k_t^{\theta} && (37) \\ g_{t+1} &= \frac{k_{t+1}^{\alpha}}{k_t^{\alpha}}(x_{t+1})^{\frac{\gamma}{1-\alpha}}, && (38) \end{aligned}$$

where $\gamma < 1-\alpha$ or $\gamma = 1-\alpha$. The two possible growth rates are

$$\begin{aligned} \bar{g}_1 &= [B(1-u)]^{\frac{\gamma}{1-\alpha}} && (39) \\ \bar{g}_2 &= [B(1-u)] && (40) \end{aligned}$$

Without ambiguity, $\bar{g}_2 > \bar{g}_1$. This is not surprising as the returns to scale in the human capital sector cannot be increasing.

6 Conclusion

This paper provides the technological conditions for the highest income growth rate in an OLG model in which the returns to scale may be non constant in both the output and the education sectors. This condition may apply to situations where increasing returns are compatible with perfect competition. Our result shows that many levels of steady state growth rates are possible depending on the allocations of investment goods to both sectors and their income shares. This may contribute to accounting for the growth differentials observed across countries and why some grow faster than others.

This paper follows Rebelo (1991) by assuming constant fractions v and u of physical and human capital allocated to the two sectors. It is nonetheless possible to endogenize u and v by considering, for instance, parental or private funding of education. They are simply particular applications of our general setup.

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A Proof of Proposition 1

Let us use the growth rate of regime 1 to prove proposition 2 without loss of generality. This growth rate is:

$$\bar{g}_1 = [B(1-v)^{\theta}(1-u)^{\eta}]^{\frac{\gamma}{2-\alpha-\eta}} \left(\frac{\beta(1-\alpha)Av^{\alpha}u^{\gamma}L^{\mu}}{1+\beta}\right)^{\frac{1-\eta}{2-\alpha-\eta}} \tag{41}$$

Let us define $\gamma = \phi(1-\alpha)$ and $\theta = \frac{1-\eta}{\phi}$, where ϕ is a strictly positive constant which is strictly less than one in growth regime 1 and equal to one in growth regime 2. Then $\bar{g}_1$ is a function of u, v, α, and η. Thus,

$$\bar{g}_1 = [B(1-v)^{\frac{1-\eta}{\phi}}(1-u)^{\eta}]^{\frac{\phi(1-\alpha)}{2-\alpha-\eta}} \left(\frac{\beta(1-\alpha)Av^{\alpha}u^{\phi(1-\alpha)}L^{\mu}}{1+\beta}\right)^{\frac{1-\eta}{2-\alpha-\eta}} \tag{42}$$

The growth rate $\bar{g}_1$ reaches a local maximum if the three following conditions are met:

i) the first derivatives with respect to u and v are equal to zero evaluated at the critical point;

ii) the difference between the square of the cross partial derivatives and the product of the second partial derivatives evaluated at the critical point is negative;

iii) the second partial derivative with respect to u is negative.

i) The first derivative of $\bar{g}_1$ with respect to u:

$$\begin{aligned}\frac{\partial \bar{g}_1}{\partial u} &= -\frac{\eta\phi(1-\alpha)}{2-\alpha-\eta}(1-u)^{\frac{\eta\phi(1-\alpha)}{2-\alpha-\eta}-1}[B(1-v)^{\frac{1-\eta}{\phi}}]^{\frac{\phi(1-\alpha)}{2-\alpha-\eta}} \left(\frac{\beta(1-\alpha)Av^{\alpha}u^{\phi(1-\alpha)}L^{\mu}}{1+\beta}\right)^{\frac{1-\eta}{2-\alpha-\eta}} \\ &+ \frac{\phi(1-\alpha)(1-\eta)}{2-\alpha-\eta}u^{\frac{\phi(1-\alpha)(1-\eta)}{2-\alpha-\eta}-1}[B(1-v)^{\frac{1-\eta}{\phi}}(1-u)^{\eta}]^{\frac{\phi(1-\alpha)}{2-\alpha-\eta}} \left(\frac{\beta(1-\alpha)Av^{\alpha}L^{\mu}}{1+\beta}\right)^{\frac{1-\eta}{2-\alpha-\eta}}\end{aligned} \tag{43}$$

i.e.,

$$\begin{aligned}\frac{\partial \bar{g}_1}{\partial u} &= [B(1-v)^{\frac{1-\eta}{\phi}}]^{\frac{\phi(1-\alpha)}{2-\alpha-\eta}} \left(\frac{\beta(1-\alpha)Av^{\alpha}L^{\mu}}{1+\beta}\right)^{\frac{1-\eta}{2-\alpha-\eta}} \\ &\left\{\frac{\phi(1-\alpha)}{2-\alpha-\eta}u^{\frac{\phi(1-\alpha)(1-\eta)}{2-\alpha-\eta}}(1-u)^{\frac{\eta\phi(1-\alpha)}{2-\alpha-\eta}}\right\}\left(\frac{1-\eta}{u} - \frac{\eta}{1-u}\right)\end{aligned} \tag{44}$$

$$\frac{\partial \bar{g}_1}{\partial u} = 0 \text{ if and only if } \frac{u}{1-u} = \frac{1-\eta}{\eta}.$$

The first derivative of $\bar{g}_1$ with respect to v:

$$\begin{aligned}
\frac{\partial \bar{g}_1}{\partial v} &= -\frac{(1-\alpha)(1-\eta)}{2-\alpha-\eta}(1-v)^{\frac{(1-\alpha)(1-\eta)}{2-\alpha-\eta}-1}[B(1-u)^{\eta}]^{\frac{\phi(1-\alpha)}{2-\alpha-\eta}}\left(\frac{\beta(1-\alpha)Av^{\alpha}u^{\phi(1-\alpha)}L^{\mu}}{1+\beta}\right)^{\frac{1-\eta}{2-\alpha-\eta}} \\
&+ \frac{\alpha(1-\eta)}{2-\alpha-\eta}v^{\frac{\alpha(1-\eta)}{2-\alpha-\eta}-1}[B(1-v)^{\frac{1-\eta}{\phi}}(1-u)^{\eta}]^{\frac{\phi(1-\alpha)}{2-\alpha-\eta}}\left(\frac{\beta(1-\alpha)Au^{\phi(1-\alpha)}L^{\mu}}{1+\beta}\right)^{\frac{1-\eta}{2-\alpha-\eta}} \qquad (45)
\end{aligned}$$

i.e.,

$$\begin{aligned}
\frac{\partial \bar{g}_1}{\partial v} &= [B(1-u)^{\eta}]^{\frac{\phi(1-\alpha)}{2-\alpha-\eta}}\left(\frac{\beta(1-\alpha)Au^{\phi(1-\alpha)}L^{\mu}}{1+\beta}\right)^{\frac{1-\eta}{2-\alpha-\eta}} \\
&\left\{\frac{1-\eta}{2-\alpha-\eta}v^{\frac{\alpha(1-\eta)}{2-\alpha-\eta}}(1-v)^{\frac{(1-\alpha)(1-\eta)}{2-\alpha-\eta}}\right\}\left(\frac{\alpha}{v}-\frac{1-\alpha}{1-v}\right) \qquad (46)
\end{aligned}$$

$$\frac{\partial \bar{g}_1}{\partial v} = 0 \text{ if and only if } \frac{v}{1-v} = \frac{\alpha}{1-\alpha}.$$

ii) The second derivative of $\bar{g}_1$ with respect to u:

$$\begin{aligned}
\frac{\partial^2 \bar{g}_1}{\partial u^2} &= \left(\frac{\eta\phi(1-\alpha)}{2-\alpha-\eta}-1\right)\left(\frac{\eta\phi(1-\alpha)}{2-\alpha-\eta}\right)(1-u)^{\frac{\eta\phi(1-\alpha)}{2-\alpha-\eta}-2} \\
&[B(1-v)^{\frac{1-\eta}{\phi}}]^{\frac{\phi(1-\alpha)}{2-\alpha-\eta}}\left(\frac{\beta(1-\alpha)Av^{\alpha}u^{\phi(1-\alpha)}L^{\mu}}{1+\beta}\right)^{\frac{1-\eta}{2-\alpha-\eta}} \\
&+ \left(\frac{\phi(1-\alpha)(1-\eta)}{2-\alpha-\eta}-1\right)\left(\frac{\phi(1-\alpha)(1-\eta)}{2-\alpha-\eta}\right)u^{\frac{\phi(1-\alpha)(1-\eta)}{2-\alpha-\eta}-2} \\
&[B(1-v)^{\frac{1-\eta}{\phi}}(1-u)^{\eta}]^{\frac{\phi(1-\alpha)}{2-\alpha-\eta}}\left(\frac{\beta(1-\alpha)Av^{\alpha}L^{\mu}}{1+\beta}\right)^{\frac{1-\eta}{2-\alpha-\eta}} \qquad (47)
\end{aligned}$$

i.e.,

$$\begin{aligned}
\frac{\partial^2 \bar{g}_1}{\partial u^2} &= [B(1-v)^{\frac{1-\eta}{\phi}}]^{\frac{\phi(1-\alpha)}{2-\alpha-\eta}}\left(\frac{\beta(1-\alpha)Av^{\alpha}L^{\mu}}{1+\beta}\right)^{\frac{1-\eta}{2-\alpha-\eta}} \\
&\left\{\frac{\phi(1-\alpha)}{(2-\alpha-\eta)^2}u^{\frac{\phi(1-\alpha)(1-\eta)}{2-\alpha-\eta}}(1-u)^{\frac{\eta\phi(1-\alpha)}{2-\alpha-\eta}}\right\} \\
&\left(\frac{(1-\eta)[\phi(1-\alpha)(1-\eta)-2+\alpha+\eta]}{u^2}+\frac{\eta[\phi(1-\alpha)\eta-2+\alpha+\eta]}{(1-u)^2}\right) \qquad (48)
\end{aligned}$$

i.e.,

$$\begin{aligned}
\frac{\partial^2 \bar{g}_1}{\partial u^2} &= [B(1-v)^{\frac{1-\eta}{\phi}}]^{\frac{\phi(1-\alpha)}{2-\alpha-\eta}} \left(\frac{\beta(1-\alpha)Av^{\alpha}L^{\mu}}{1+\beta}\right)^{\frac{1-\eta}{2-\alpha-\eta}} \\
&\left\{\frac{\phi(1-\alpha)}{(2-\alpha-\eta)^2}u^{\frac{\phi(1-\alpha)(1-\eta)}{2-\alpha-\eta}}(1-u)^{\frac{\eta\phi(1-\alpha)}{2-\alpha-\eta}}\right\} \\
&\left(\frac{(1-\eta)[-(1-\alpha)(1-\eta)(1-\phi)-(1-\alpha\eta)]}{u^2}+\frac{\eta[-(1-\alpha)(1-\phi\eta)-(1-\eta)]}{(1-u)^2}\right)
\end{aligned} \tag{49}$$

Clearly, $\frac{\partial^2 \bar{g}_1}{\partial u^2} < 0$.

The second derivative of $\bar{g}_1$ with respect to v:

$$\begin{aligned}
\frac{\partial^2 \bar{g}_1}{\partial v^2} &= \left(\frac{(1-\alpha)(1-\eta)}{2-\alpha-\eta}-1\right)\left(\frac{(1-\alpha)(1-\eta)}{2-\alpha-\eta}\right)(1-v)^{\frac{(1-\alpha)(1-\eta)}{2-\alpha-\eta}-2} \\
&\quad [B(1-u)^{\eta}]^{\frac{\phi(1-\alpha)}{2-\alpha-\eta}}\left(\frac{\beta(1-\alpha)Av^{\alpha}u^{\phi(1-\alpha)}L^{\mu}}{1+\beta}\right)^{\frac{1-\eta}{2-\alpha-\eta}} \\
&+ \left(\frac{\alpha(1-\eta)}{2-\alpha-\eta}-1\right)\left(\frac{\alpha(1-\eta)}{2-\alpha-\eta}\right)v^{\frac{\alpha(1-\eta)}{2-\alpha-\eta}-2} \\
&\quad [B(1-v)^{\frac{1-\eta}{\phi}}(1-u)^{\eta}]^{\frac{\phi(1-\alpha)}{2-\alpha-\eta}}\left(\frac{\beta(1-\alpha)Au^{\phi(1-\alpha)}L^{\mu}}{1+\beta}\right)^{\frac{1-\eta}{2-\alpha-\eta}}
\end{aligned} \tag{50}$$

i.e.,

$$\begin{aligned}
\frac{\partial^2 \bar{g}_1}{\partial v^2} &= [B(1-u)^{\eta}]^{\frac{\phi(1-\alpha)}{2-\alpha-\eta}}\left(\frac{\beta(1-\alpha)Au^{\phi(1-\alpha)}L^{\mu}}{1+\beta}\right)^{\frac{1-\eta}{2-\alpha-\eta}} \\
&\left\{\frac{1-\eta}{(2-\alpha-\eta)^2}v^{\frac{\alpha(1-\eta)}{2-\alpha-\eta}}(1-v)^{\frac{(1-\alpha)(1-\eta)}{2-\alpha-\eta}}\right\} \\
&\left(\frac{\alpha[\alpha(1-\eta)-2+\alpha+\eta]}{v^2}+\frac{(1-\alpha)[(1-\alpha)(1-\eta)-2+\alpha+\eta]}{(1-v)^2}\right)
\end{aligned} \tag{51}$$

i.e.,

$$\begin{aligned}
\frac{\partial^2 \bar{g}_1}{\partial v^2} &= [B(1-u)^{\eta}]^{\frac{\phi(1-\alpha)}{2-\alpha-\eta}}\left(\frac{\beta(1-\alpha)Au^{\phi(1-\alpha)}L^{\mu}}{1+\beta}\right)^{\frac{1-\eta}{2-\alpha-\eta}} \\
&\left\{\frac{1-\eta}{(2-\alpha-\eta)^2}v^{\frac{\alpha(1-\eta)}{2-\alpha-\eta}}(1-v)^{\frac{(1-\alpha)(1-\eta)}{2-\alpha-\eta}}\right\} \\
&\left(\frac{\alpha[-(1-\eta)(1-\alpha)-(1-\alpha)]}{v^2}+\frac{(1-\alpha)[-\eta(1-\alpha)-(1-\eta)]}{(1-v)^2}\right)
\end{aligned} \tag{52}$$

Clearly, $\frac{\partial^2 \bar{g}_1}{\partial v^2} < 0$.

The cross partial derivatives:

$$\begin{aligned}
\frac{\partial^2 \bar{g}_1}{\partial u \partial v} &= [B(1-v)^{\frac{1-\eta}{\phi}}]^{\frac{\phi(1-\alpha)}{2-\alpha-\eta}} \left(\frac{\beta(1-\alpha)Av^{\alpha}L^{\mu}}{1+\beta}\right)^{\frac{1-\eta}{2-\alpha-\eta}} \\
& \left\{\frac{\phi(1-\alpha)}{2-\alpha-\eta} u^{\frac{\phi(1-\alpha)(1-\eta)}{2-\alpha-\eta}} (1-u)^{\frac{\eta\phi(1-\alpha)}{2-\alpha-\eta}}\right\} \left(\frac{1-\eta}{u} - \frac{\eta}{1-u}\right) \\
& \left(\frac{1-\eta}{2-\alpha-\eta}\right) \left(\frac{-(1-\alpha)}{1-v} + \frac{\alpha}{v}\right) \qquad (53)
\end{aligned}$$

If $v = \alpha$ the last term in parentheses is equal to 0 then $\frac{\partial^2 \bar{g}_1}{\partial u \partial v} = 0$

$$\begin{aligned}
\frac{\partial^2 \bar{g}_1}{\partial v \partial u} &= [B(1-u)^{\eta}]^{\frac{\phi(1-\alpha)}{2-\alpha-\eta}} \left(\frac{\beta(1-\alpha)Au^{\phi(1-\alpha)}L^{\mu}}{1+\beta}\right)^{\frac{1-\eta}{2-\alpha-\eta}} \\
& \left\{\frac{1-\eta}{(2-\alpha-\eta)^2} v^{\frac{\alpha(1-\eta)}{2-\alpha-\eta}} (1-v)^{\frac{(1-\alpha)(1-\eta)}{2-\alpha-\eta}}\right\} \\
& \left(\frac{\alpha[-(1-\alpha)(2-\eta)]}{v^2} + \frac{(1-\alpha)[\alpha\eta - 1]}{(1-v)^2}\right) \\
& \left(\frac{\phi(1-\alpha)}{2-\alpha-\eta}\right) \left(\frac{-\eta}{1-u} + \frac{1-\eta}{u}\right) \qquad (54)
\end{aligned}$$

If $u = 1-\eta$ the last term in parentheses is equal to 0 then $\frac{\partial^2 \bar{g}_1}{\partial v \partial u} = 0$.
Let us denote

$$r \equiv \frac{\partial^2 \bar{g}_1}{\partial u^2}(1-\eta, \alpha) < 0 \qquad (55)$$

$$s \equiv \frac{\partial^2 \bar{g}_1}{\partial u \partial v}(1-\eta, \alpha) = 0 \qquad (56)$$

$$t \equiv \frac{\partial^2 \bar{g}_1}{\partial u^2}(1-\eta, \alpha) < 0 \qquad (57)$$

After our calculations, we can conclude that

$$s^2 - rt < 0 \qquad (58)$$

iii) Finally, our computations lead to

$$r < 0 \qquad (59)$$

Therefore, the function $\bar{g}_1(u, v)$ admits a maximum on the domain [0,1] at the point (u, v) where $u = 1-\eta$ and $v = \alpha$ which verifies condition (12). For growth regime 2, as $\theta = 1-\eta$, the condition becomes $u = \theta$ and $v = \alpha$.

B Proof of Proposition 2

The growth rate of regime 1, $\bar{g}_3$, evaluated at the maximum $(u = 1-\eta, v = \alpha)$ is:

$$\bar{g}_1 = [B(1-\alpha)^{\theta}(1-\phi\theta)^{1-\phi\theta}]^{\frac{\phi(1-\alpha)}{1-\alpha-\phi\theta}} \left(\frac{\beta(1-\alpha)A\alpha^{\alpha}(\phi\theta)^{\phi(1-\alpha)}L^{\mu}}{1+\beta}\right)^{\frac{\phi\theta}{1-\alpha-\phi\theta}}, \quad (60)$$

where $\phi = \frac{\gamma}{1-\alpha} = \frac{1-\eta}{\theta}$. The growth rate of regime 2, $\bar{g}_3$, evaluated at the maximum $(u = \theta, v = \alpha)$ is:

$$\bar{g}_2 = [B(1-\alpha)^{\theta}(1-\theta)^{1-\theta}]^{\frac{(1-\alpha)}{1-\alpha-\theta}} \left(\frac{\beta(1-\alpha)A\alpha^{\alpha}\theta^{(1-\alpha)}L^{\mu}}{1+\beta}\right)^{\frac{\theta}{1-\alpha-\theta}}. \quad (61)$$

The difference between (60) and (61) is that $\bar{g}_1$ is a function of ϕ. In fact, $\phi = 1$ in $\bar{g}_2$ while $\phi < 1$ in $\bar{g}_1$. To prove that $\bar{g}_1 > \bar{g}_2$, it suffices to show that the derivative of $\bar{g}_1$ with respect to ϕ is negative.

Let us apply the logarithm to (60):

$$\begin{aligned}\ln(\bar{g}_1) &= \frac{\phi(1-\alpha)}{1-\alpha-\phi\theta}\ln(B) + \frac{\phi\theta(1-\alpha)}{1-\alpha-\phi\theta}\ln(1-\alpha) + \frac{\phi(1-\phi\theta)(1-\alpha)}{1-\alpha-\phi\theta}\ln(1-\phi\theta)\\ &+\frac{\phi\theta}{1-\alpha-\phi\theta}\ln\left(\frac{\beta(1-\alpha)A\alpha^{\alpha}(\phi\theta)^{\phi(1-\alpha)}L^{\mu}}{1+\beta}\right). \quad (62)\end{aligned}$$

The derivative of $\ln(\bar{g}_1)$ with respect ϕ is:

$$\begin{aligned}\frac{d\ln(\bar{g}_1)}{d\phi} &= \frac{1-\alpha}{(1-\alpha-\phi\theta)^2}\{(1-\alpha)\ln(B) + \theta(1-\alpha)\ln(1-\alpha)\\ &+ (1-\alpha)\ln(1-\phi\theta) + [\phi\theta(2(1-\alpha)-\phi\theta)]\{\ln(\phi\theta)-\ln(1-\phi\theta)\}\\ &+ \theta\ln(\beta(1-\alpha)A\alpha^{\alpha}L^{\mu}) - \theta\ln(1+\beta)\} \quad (63)\end{aligned}$$

To study the sign of (63), we first study the sign of one of its term: $[2(1-\alpha)-\phi\theta]\{\ln(\phi\theta)-\ln(1-\phi\theta)\}$. We can show that:

- if $\phi\theta > \frac{1}{2}$ then $[2(1-\alpha)-\phi\theta] < 0$ and $\{\ln(\phi\theta)-\ln(1-\phi\theta)\} > 0$, and the product is negative;
- if $\phi\theta < \frac{1}{2}$ then $[2(1-\alpha)-\phi\theta] > 0$ and $\{\ln(\phi\theta)-\ln(1-\phi\theta)\} < 0$, and the product is negative;
- if $\phi\theta = \frac{1}{2}$ then $\{\ln(\phi\theta)-\ln(1-\phi\theta)\} = 0$ and the product is null;

Therefore, the term $[2(1-\alpha)-\phi\theta]\{\ln(\phi\theta)-\ln(1-\phi\theta)\}$ is negative or null.

Finally, we can conclude that the sign of (63) depends on the level of B, A and L^{μ}:

- if $B^{1-\alpha}A^{\theta}L^{\theta\mu} > \left(\frac{[(1-\alpha)^{\theta}(1-\phi\theta)]^{1-\alpha}(\beta(1-\alpha)\alpha^{\alpha})^{\theta}}{1+\beta}\right)\left(\frac{\phi\theta}{1-\phi\theta}\right)^{\phi\theta[2(1-\alpha)-\phi\theta]}$ then $\frac{d\ln(\bar{g}_1)}{d\phi} > 0$. Thus $\bar{g}_1 < \bar{g}_2$.
- if $B^{1-\alpha}A^{\theta}L^{\theta\mu} < \left(\frac{[(1-\alpha)^{\theta}(1-\phi\theta)]^{1-\alpha}(\beta(1-\alpha)\alpha^{\alpha})^{\theta}}{1+\beta}\right)\left(\frac{\phi\theta}{1-\phi\theta}\right)^{\phi\theta[2(1-\alpha)-\phi\theta]}$ then $\frac{d\ln(\bar{g}_1)}{d\phi} < 0$. Thus $\bar{g}_1 > \bar{g}_2$.